

A Bayesian approach to evaluate confidence intervals in counting experiments with background

F. Loparco

*Dipartimento Interateneo di Fisica “M. Merlin”, dell’Università degli Studi di Bari and
INFN Sezione di Bari, Via Amendola 173, I-70126 Bari, Italy*

M. N. Mazziotta

INFN Sezione di Bari, Via Amendola 173, I-70126 Bari, Italy

Abstract

In this paper we propose a procedure to evaluate Bayesian confidence intervals in counting experiments where both signal and background fluctuations are described by the Poisson statistics. The results obtained when the method is applied to the calculation of upper limits will also be illustrated.

Keywords: Counting experiments, Confidence intervals, Upper limits

1. Introduction

The evaluation of confidence intervals and limits is a common task in particle physics [1]. Usually the goal of an experiment is that of determining a parameter θ starting from a set of measurements of a random variable x (the outcome of the experiment) and assuming an hypothesis for the probability distribution function (p.d.f.) $f(x|\theta)$. A confidence interval for θ at the confidence level (or coverage probability) $1 - \beta$ includes the true value of θ with a probability $1 - \beta$. This means that if the experiment were repeated many times, the estimates of θ will fall in the confidence interval in a fraction

Email addresses: loparco@ba.infn.it (F. Loparco), mazziotta@ba.infn.it (M. N. Mazziotta)

24 $1 - \beta$ of the experiments ¹.

25 Confidence intervals can be evaluated following either the frequentist [2,
26 3, 4] or the Bayesian approaches (a detailed discussion about these methods
27 can be found in ref. [1]). One of the main issues arising when applying the
28 frequentist approach is that the confidence intervals may include unphysical
29 regions for the parameter. These cases can be fixed by introducing some
30 “ad-hoc” corrections in the mathematical procedures used to evaluate the
31 intervals (see for instance ref [4, 5]). Such issues can be easily avoided when
32 following the Bayesian approach, by properly choosing the prior p.d.f. ²
33 for the parameter. However, the fact that the Bayesian approach requires
34 a choice of the prior p.d.f. for the parameter introduces some degree of
35 arbitrariness in the evaluation of confidence intervals.

36 In many cases the outcome of an experiment can be described in terms
37 of a set parameters, not all being of any interest for the final result (nuisance
38 parameters). In these cases the experimenter wishes to evaluate confidence
39 regions on the parameters of interest, in a manner that is independent on
40 the nuisance parameters. From a mathematical point of view, the Bayesian
41 approach allows one to treat the nuisance parameters in a very simple and
42 straightforward way. Indicating respectively with θ and ν the parameters of
43 interest and the nuisance parameters, one has to write down their joint prior
44 p.d.f. $\pi(\theta, \nu)$ ³ and to evaluate from it the marginal prior p.d.f. for θ . In
45 most cases θ and ν are independent random variables, and their joint p.d.f.
46 can be factorized as $\pi(\theta, \nu) = \pi(\theta)\pi(\nu)$, thus making the calculation easier.

47 In the following sections we will illustrate an application of the Bayesian
48 approach to the analysis of a counting experiment with background. We
49 will assume that the outcome of the experiment can be modeled in terms of
50 a parameter of interest (signal) and of a nuisance parameter (background),
51 that will be supposed to be independent on each other. The posterior p.d.f.

¹In the most general case θ is a vector of parameters, i.e. $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ and x represents a set of observables, i.e. $x = (x_1, x_2, \dots, x_m)$. Consequently, the p.d.f. $f(x|\theta) = f(x_1, \dots, x_m|\theta_1, \dots, \theta_n)$ will be a function of $m+n$ variables. Also, the confidence interval defined in the case of a single parameter becomes a n -dimensional confidence region for the vector of parameters θ .

²This can be done by assigning null probabilities to the values of θ in the unphysical regions.

³Hereafter we will use the greek symbol π to indicate prior p.d.f.s and the latin symbol p to indicate posterior p.d.f.s

for the signal will be derived starting from a set of simple and quite general assumptions on the prior p.d.f.s for both the signal and the background. The formulas to evaluate upper limits on the signal will also be derived and the results will be discussed. Our results are a generalization of the ones illustrated in ref. [1], where the same problem is discussed, but without taking background fluctuations into account.

2. Formulation of the problem

Let us consider a counting experiment in which one wants to measure the signal counts in presence of a background, that is measured in a subsidiary experiment. Such a situation can happen when measuring the activity of a weak radioactive source in presence of background with a counting device, like a Geiger-Muller detector. In this case two independent measurements are carried out, one for the signal and one for the background, and the results have to be properly combined. Another example is an experiment in which the signal and the background are evaluated looking at two different space regions. Also in this case, the signal is evaluated by combining the counts measured in the signal and background regions ⁴.

In the following we shall denote with n and m the number of counts measured respectively in the signal and in the background regions. We will also indicate with s and b the true values of the signal and of the background counts respectively, and we will assume that the true value of the background counts in the signal region is given by cb , where c is a constant value that is assumed to be exactly known. Let us consider, as a first example, an experiment to measure the activity of a radioactive source. In this case n and m are respectively the counts recorded during a time interval T_s in presence of the source, and during a time interval T_b when the source has been removed. The value of c can be estimated as $c = T_s/T_b$, and the source activity can be evaluated as s/T_s once the value of s has been measured. A second example, taken from the astrophysics, is the measurement of the photon flux of a point source in the sky with a gamma-ray detector. In this case the signal region can be a cone centered on the source direction with a

⁴We prefer to use the word “combine” instead of “subtract” because in many cases the signal cannot be evaluated by simply subtracting the counts in the background region from the ones in the signal region. This may happen for instance when the counts in the background region are more than the ones in the signal region.

83 given angular aperture, while the background region can be an annulus far
 84 away from the source. In this case c can be evaluated as the ratio between
 85 the solid angles of the signal region and of the background region, eventually
 86 multiplied for the ratio between the live times of the two regions.

87 Under the above assumptions, the probability of measuring m counts in
 88 the background region will be a Poisson distribution with mean value b , i.e.:

$$p(m|b) = e^{-b} \frac{b^m}{m!} \quad (1)$$

89 while the probability of measuring n counts in the signal region will be a
 90 Poisson distribution with mean value $s + cb$, i.e.:

$$p(n|s, b) = e^{-(s+cb)} \frac{(s + cb)^n}{n!} \quad (2)$$

91 Since the two measurements are independent, the joint p.d.f. for n and m
 92 will be given by:

$$p(n, m|s, b) = e^{-(s+cb)} \frac{(s + cb)^n}{n!} e^{-b} \frac{b^m}{m!} \quad (3)$$

93 Our problem is that of evaluating a Bayesian confidence interval (or
 94 an upper limit) for the parameter s , independently on b . An analogous
 95 problem is discussed in the textbook [6] and in ref. [1], where the background
 96 value is assumed to be exactly known. Our discussion will be therefore
 97 a generalization of refs. [1, 6]. A possible solution of the problem taking
 98 background fluctuations into account is given in ref. [7]. However, in ref. [7],
 99 a gaussian p.d.f. is used to model the background, the results being valid
 100 in the case of large counts. A similar problem is also illustrated in ref. [8],
 101 where Bayesian confidence interval are evaluated for a Poisson signal with
 102 known background and with fluctuations on the detection efficiency.

103 Another possible solution to our problem is also given in refs. [4, 5], where
 104 a frequentist approach is followed with the application of the profile likelihood
 105 method. However, while the procedure described in refs. [4, 5] requires some
 106 adjustments to handle the cases when $n < cm$ or when either $n = 0$ or
 107 $m = 0$, in our method the treatment of these cases is straightforward and
 108 does not require any adjustment. In fact, the formulas that will be derived
 109 in sections 3 and 4 are valid for all the values of n and m .

110 3. The Bayesian approach

111 The implementation of the Bayesian approach requires the “probabilistic
112 inversion” of eq. 3, i.e. the evaluation of the conditional p.d.f. $p(s, b|n, m)$
113 starting from $p(n, m|s, b)$ and applying the Bayes’theorem.

114 3.1. Choice of the priors

115 The application of the Bayes’theorem starts from the assumption of a
116 prior p.d.f. for both the random variables s and b . In the following we will
117 assume that s and b are independent.

118 For the true background value b we will assume a uniform prior:

$$\pi(b) = \begin{cases} \pi_0 & b \geq 0 \\ 0 & b < 0 \end{cases} \quad (4)$$

119 where $\pi_0 > 0$ is a constant. The parameter b is only constrained to be
120 non-negative.

121 On the other hand, for the signal true value s we will assume a prior p.d.f.
122 given by:

$$\pi(s) = \begin{cases} ks^{-\alpha} & s \geq 0 \\ 0 & s < 0 \end{cases} \quad (5)$$

123 with $k > 0$. Also in this case, the only constraint on the parameter s is that
124 it must be non-negative.

125 It is worth to point out at this stage that both $\pi(b)$ and $\pi(s)$ defined in
126 eqs. 4 and 5 are improper priors ⁵ and then they can lead to posterior p.d.f.s
127 that are not normalizable. In particular, as it will be shown in sec. 3.3, this
128 happens when setting $\alpha \geq 1$. Our calculations will therefore be valid only
129 for $\alpha < 1$.

130 The uniform prior for the signal is obtained by setting $\alpha = 0$ and
131 represents the natural choice when the experimenter does not have any model
132 for the signal. On the other hand, the choice of a power-law prior with $\alpha > 0$
133 will reflect the experimenter’s belief that small signal values are more likely
134 than larger ones. It is also possible to choose negative values of α : this choice,
135 that is rather uncommon, would favour larger signal values with respect to
136 smaller ones ⁶.

⁵It’s easy to show that $\int_0^\infty \pi(b)db = \infty$ and $\int_0^\infty \pi(s)ds = \infty$ for any value of α .

⁶Let $s_1 < s_2$ be two possible signal values and let us consider the intervals $[s_1, s_1 + \Delta s]$

137 *3.2. Evaluation of the background posterior p.d.f.*

138 Applying the Bayes'theorem and using for $\pi(b)$ the expression in eq. 4 it
 139 is possible to obtain the following equation:

$$p(b|m) = \frac{p(m|b)\pi(b)}{\int p(m|b)\pi(b)db} = \frac{p(m|b)}{\int_0^\infty p(m|b)db} \quad (6)$$

140 Finally, replacing $p(m|b)$ with its expression given in eq. 1, it is
 141 straightforward to obtain the final result:

$$p(b|m) = e^{-b} \frac{b^m}{m!} \quad (7)$$

142 Note that even though the expression of $p(b|m)$ in eq. 7 is the same as
 143 that of $p(m|b)$ in eq. 1, their meanings are completely different. In fact, the
 144 random variable in eq. 1 is m , and the formula tells that m follows a Poisson
 145 distribution; on the other hand, the random variable in eq. 7 is b , and the
 146 formula tells that b follows a Gamma distribution.

147 Using eq. 7 one can also easily evaluate the average value of b and its
 148 standard deviation. It is easy to show that $\langle b \rangle = m + 1$ and $\sigma_b = \sqrt{m + 1}$.

149 *3.3. Evaluation of the signal posterior p.d.f.*

150 The Bayes'theorem can be applied to get the joint posterior p.d.f. for
 151 both s and b :

$$p(s, b|n, m) = \frac{p(n, m|s, b)\pi(s)\pi(b)}{\int ds \int db p(n, m|s, b)\pi(s)\pi(b)} \quad (8)$$

152 Replacing $p(n, m|s, b)$ with its expression given in eq. 3 and $\pi(b)$ and
 153 $\pi(s)$ with their expressions given in eqs. 1 and 2 respectively, eq. 8 can be
 154 rewritten as follows:

$$p(s, b|n, m) = \frac{e^{-s-(c+1)b} b^m s^{-\alpha} (s + cb)^n}{\int_0^\infty ds \int_0^\infty db e^{-s-(c+1)b} b^m s^{-\alpha} (s + cb)^n} \quad (9)$$

and $[s_2, s_2 + \Delta s]$. The ratio between the probabilities of finding s in the two intervals is given by $R = P(s_1 < s < s_1 + \Delta s)/P(s_2 < s < s_2 + \Delta s) = (s_2/s_1)^\alpha$. When $\alpha > 0$ ($\alpha < 0$) then $R > 1$ ($R < 1$) and small (large) signal values are more likely than larger (smaller) values. For a discussion about the choice of priors in the Bayesian approach see for instance the textbook [6].

155 Indicating with N the denominator in the right-hand side of eq. 9, it can
 156 be rewritten as:

$$N = \int_0^\infty db b^m e^{-(c+1)b} \int_0^\infty ds e^{-s} s^{-\alpha} (s + cb)^n = \int_0^\infty db b^m e^{-(c+1)b} f(b) \quad (10)$$

157 where we have indicated with $f(b)$ the result of the integral in ds , that can
 158 be seen as a function of the variable b .

159 Applying the binomial theorem, the term $(s + cb)^n$ in the expression of
 160 $f(b)$ can be expanded as follows:

$$(s + cb)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} s^k (cb)^{n-k} \quad (11)$$

161 Using this result, the expression of $f(b)$ becomes:

$$f(b) = n! \sum_{k=0}^n \frac{(cb)^{n-k}}{k!(n-k)!} \int_0^\infty e^{-s} s^{k-\alpha} ds \quad (12)$$

162 and, taking into account the definition of the Gamma function ⁷ the previous
 163 equation can be written as:

$$f(b) = \Gamma(n+1) \sum_{k=0}^n (cb)^{n-k} \frac{\Gamma(k-\alpha+1)}{\Gamma(k+1)\Gamma(n-k+1)} \quad (13)$$

164 Introducing the expression of $f(b)$ given by eq. 13 in the expression of N
 165 given by eq. 10 we get the following result:

$$N = \Gamma(n+1) \sum_{k=0}^n \frac{\Gamma(k-\alpha+1)c^{n-k}}{\Gamma(k+1)\Gamma(n-k+1)} \int_0^\infty db b^{m+n-k} e^{-(c+1)b} \quad (14)$$

166 By making a proper change of variable, the integral in the right-hand side
 167 eq. 14 can be expressed in terms of a Gamma function. Hence eq. 14 can be
 168 rewritten as follows:

⁷The Gamma function is defined as $\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1}$. It can be shown that $\Gamma(z) = (z-1)!$ when z is a positive integer.

$$N = \frac{\Gamma(n+1)}{(c+1)^{m+1}} \sum_{k=0}^n \frac{\Gamma(k-\alpha+1)\Gamma(m+n-k+1)}{\Gamma(k+1)\Gamma(n-k+1)} \left(\frac{c}{c+1}\right)^{n-k} \quad (15)$$

169 The joint posterior p.d.f. for s and b is then given by:

$$p(s, b|n, m) = \frac{1}{N} e^{-s-(c+1)b} b^m s^{-\alpha} (s+cb)^n \quad (16)$$

170 with the expression of N given in eq. 15.

171 To evaluate the marginal p.d.f. for s we need to integrate the joint p.d.f.
172 with respect to b :

$$p(s|n, m) = \int_0^\infty p(s, b|n, m) db = \frac{1}{N} e^{-s} s^{-\alpha} \int_0^\infty db e^{-(c+1)b} b^m (s+cb)^n \quad (17)$$

173 Indicating with $g(s)$ the integral in the right-hand side of eq. 17, it can
174 be evaluated in a similar way to that used to calculate $f(b)$. It is easy to
175 show that:

$$g(s) = \frac{\Gamma(n+1)}{(c+1)^{m+1}} \sum_{k=0}^n \frac{\Gamma(m+n-k+1)}{\Gamma(k+1)\Gamma(n-k+1)} \left(\frac{c}{c+1}\right)^{n-k} s^k \quad (18)$$

176 Introducing in eq. 17 the expression of $g(s)$ given by eq. 18 and the
177 expression of N given by eq. 15, it is possible to show that the posterior
178 p.d.f. for s is given by:

$$p(s|n, m) = \frac{\sum_{k=0}^n \frac{\Gamma(m+n-k+1)}{\Gamma(k+1)\Gamma(n-k+1)} \left(\frac{c}{c+1}\right)^{n-k} s^{k-\alpha} e^{-s}}{\sum_{k=0}^n \frac{\Gamma(k-\alpha+1)\Gamma(m+n-k+1)}{\Gamma(k+1)\Gamma(n-k+1)} \left(\frac{c}{c+1}\right)^{n-k}} \quad (19)$$

179 It is worth to point out here that eq. 19 is valid only for $\alpha < 1$. In fact,
180 if $\alpha \geq 1$, the argument $k - \alpha + 1$ of the Gamma function in the sum in the
181 right-hand side of the denominator will be null or negative, and consequently
182 the Gamma function either will assume negative values or will diverge. As
183 discussed in sec. 3.1, this behaviour is due to the fact that since the posterior

184 p.d.f. $p(s|n, m)$ is obtained starting from improper priors, its normalization
 185 is not guaranteed.

186 Fig. 1 shows the posterior p.d.f.s for the signal s evaluated for some
 187 different values of n and m in the case $c = 1$, i.e. when the background
 188 region has the same size as the signal region. The calculations have been
 189 performed in the cases $\alpha = 0.5$ (small signal expected), $\alpha = 0$ (uniform
 190 prior) and $\alpha = -0.5$ (large signal expected). The differences between the
 191 three p.d.f.s become negligible when n is larger than cm .

192 From eq. 19 one can also easily calculate the moments of the p.d.f.
 193 $p(s|n, m)$. It's easy to show that the h th moment of the p.d.f. is given
 194 by:

$$\langle s^h | n, m \rangle = \frac{\sum_{k=0}^n \frac{\Gamma(k - \alpha + 1 + h) \Gamma(m + n - k + 1)}{\Gamma(k + 1) \Gamma(n - k + 1)} \left(\frac{c}{c + 1} \right)^{n-k}}{\sum_{k=0}^n \frac{\Gamma(k - \alpha + 1) \Gamma(m + n - k + 1)}{\Gamma(k + 1) \Gamma(n - k + 1)} \left(\frac{c}{c + 1} \right)^{n-k}} \quad (20)$$

195 In particular, eq. 20 allows to calculate the expectation value of s , i.e. $\langle s \rangle$
 196 and its variance, that can be evaluated as $\text{var}(s) = \langle s^2 \rangle - \langle s \rangle^2$.

197 Finally, we can consider the limiting case with absence of background,
 198 that can be obtained by setting $c = 0$ (background region with null size). In
 199 this limit, all the terms with $[c/(c + 1)]^{n-k}$ in both the summations of eq. 19
 200 will vanish, but the ones with $k = n$, where $[c/(c + 1)]^{n-k} = 1$. Hence, in this
 201 case the posterior p.d.f. for the signal simplifies to:

$$p(s|n, m) = \frac{s^{n-\alpha} e^{-s}}{\Gamma(n - \alpha + 1)} \quad (21)$$

202 and, as expected, does not depend on m . In particular, in the case $\alpha = 0$
 203 the posterior p.d.f. for the signal becomes a Gamma distribution.

204 4. Evaluation of the upper limits on the signal

205 The posterior p.d.f. for the signal given by eq. 19 can be used to evaluate
 206 Bayesian confidence intervals for s . In particular, in this section, we will
 207 apply the result of eq. 19 to the calculation of upper limits on s .

208 To evaluate the upper limit s_u at the confidence level $1 - \beta$ we have to
 209 solve the integral equation:

$$\int_0^{s_u} p(s|n, m) ds = 1 - \beta \quad (22)$$

210 Taking advantage of the fact that

$$\int_0^{s_u} s^{k-\alpha} e^{-s} ds = \gamma(k - \alpha + 1, s_u) \quad (23)$$

211 where we have indicated with γ the incomplete Gamma function ⁸, eq. 22
 212 can be rewritten as:

$$1 - \beta = \frac{\sum_{k=0}^n \frac{\gamma(k - \alpha + 1, s_u) \Gamma(m + n - k + 1)}{\Gamma(k + 1) \Gamma(n - k + 1)} \left(\frac{c}{c + 1} \right)^{n-k}}{\sum_{k=0}^n \frac{\Gamma(k - \alpha + 1) \Gamma(m + n - k + 1)}{\Gamma(k + 1) \Gamma(n - k + 1)} \left(\frac{c}{c + 1} \right)^{n-k}} \quad (24)$$

213 Eq. 24 can be solved numerically and allows to obtain the Bayesian upper
 214 limits on s at the confidence level $1 - \beta$ for any values of n and m , given
 215 the values of c and α . We have performed our calculations using the CERN
 216 ROOT package [9]. In particular, we have implemented a code that allows
 217 to evaluate the numerical solutions s_u of eq. 24 for any given value of β with
 218 the Brent's method, using the ROOT built-in tools.

219 In fig. 2 the upper limits on the signal at 90% confidence level are shown
 220 as a function of the observed counts in the signal (n) and background (m)
 221 regions in the case $c = 1$ for three different values of α . As expected, the
 222 choice of the power law index α in the signal prior p.d.f. will affect the result
 223 on the upper limits. In particular, the upper limits on the signal will increase
 224 with decreasing α . This result is in agreement with the fact that when α is
 225 positive and close to 1 small signal values are expected while, on the other
 226 hand, when α is negative, large signal values are expected. As pointed out
 227 in sec. 1, this is a general issue of the Bayesian approach, the results being
 228 influenced by the initial belief of the experimenter.

⁸The incomplete Gamma function is defined as $\gamma(z, x) = \int_0^x dt e^{-t} t^{z-1}$. According to this definition $\gamma(z, \infty) = \Gamma(z)$.

Fig. 3 shows the upper limits on the signal at 90% confidence level as a function of the background events m for several values of the signal events n . The calculation has been performed in the case $c = 1$ with the uniform prior ($\alpha = 0$). For any given value of n , the upper limit on the signal decreases with increasing m , in agreement with the fact that (in the cases when $n > cm$) a rough estimate of s is given by $n - cm$ and the upper limit is expected to be proportional to $n - cm$. It has also to be pointed out that in the case $n = 0$, i.e. when no events are observed in the signal region, the upper limit is always equal to 2.30, independently on m ⁹.

Finally, fig. 4 shows the upper limits on the signal at 90% confidence level as a function of the signal events m for several values of the background events m . Again, the calculation has been performed in the case $c = 1$ with the uniform prior ($\alpha = 0$). The upper limits increase with increasing n , and the trend of the curves becomes almost linear when $n > cm$.

4.1. Study of the frequentist coverage

To study the frequentist coverage of the upper limits obtained with our procedure we implemented a dedicated simulation. For simplicity we studied only the case with $c = 1$, when the signal and the background regions have the same sizes.

For any given pair of values of s and b , a sample of 10^5 events was simulated, each corresponding to the outcome of an experiment. Each event consists of a couple of random integer numbers (n, m) , representing respectively the counts in the signal and in the background regions, that are generated according to the p.d.f. of eq. 3. For each couple of counts (n, m) the corresponding upper limits on s at 90% confidence level were evaluated by solving eq. 24 for different values of α . The coverage was then evaluated as the fraction of events with an associated upper limits less than the true value of the signal s .

Fig. 5 shows, as an example, the results obtained with a simulated sample of events with $s = 3.5$ and $b = 2$. The distributions of the upper limits at 90% confidence level are shown for $\alpha = 0.5$, $\alpha = 0$ and $\alpha = -0.5$. As expected,

⁹It is evident that when no events (or a few events) are observed in the signal region and a large number of events are observed in the background region the prior p.d.f. given by eq. 5 is not adequate. In these cases a different prior p.d.f. should be used, that allows for negative values of s .

the coverage increases with decreasing α , since lower values of α correspond to more conservative upper limits.

We have studied the frequentist coverage of our upper limits as a function of both s and b . The results are summarized in Fig. 6, where the coverage is plotted as a function of s in the cases when $b = 0, b = 1, b = 1.5, b = 2, b = 5$ and $b = 10$. As expected, the frequentist coverage decreases with increasing α . The choice of a uniform prior for the signal, i.e. $\alpha = 0$ guarantees a coverage that is larger than 90% for low signal values, and oscillates around 90% with increasing s . On the other hand, the choice of a less conservative prior with $\alpha = 0.5$ does not seem to affect significantly the coverage in case of an high background level.

5. Conclusions

We have developed a procedure that, following the Bayesian approach, allows to evaluate confidence intervals on the signal in experiments with background where both signal and background are modeled by the Poisson statistics. The implementation of the method is quite simple from the mathematical point of view, and does not require any adjustments to treat the cases when less events are observed in the signal region than those in the background region. The results obtained when our procedure is applied to the calculation of upper limits have been also discussed. The frequentist coverage of the upper limits evaluated with this procedure has been also studied.

References

- [1] G. Cowan et al. (Particle Data Group), *Journ. Phys. G* **37** (2010), 075021 see also <http://pdg.lbl.gov/2010/reviews/rpp2010-rev-statistics.pdf>
- [2] J. Neyman, *Phil. Trans. Royal Soc. London, Series A* **236**, 333 (1937)
- [3] G. J. Feldman, R. D. Cousins, “Unified approach to the classical statistical analysis of small signals” , *Phys. Rev. D* **57** (1998), 3873
- [4] W. A. Rolke, A. M. Lopez, “Confidence intervals and upper bounds for small signals in presence of background noise”, *Nucl. Inst. Meth.* **A458** (2000), 745

- 292 [5] W. A. Rolke, A. M. Lopez, J. Conrad, “Limits on Confidence Intervals
293 in the Presence of Nuisance Parameters”, *ArXiv:physics/0403059* (2009)
- 294 [6] G. Cowan, “Statistical Data Analysis”, Oxford University Press (1998),
295 ISBN 0-19-850155-2
- 296 [7] O. Helene, “Determination of the upper limit of a peak area”, *Nucl.*
297 *Inst. Meth.* **212** (1983), 319
- 298 [8] J. Heinrich et al., “Interval estimation in the presence of nuisance
299 parameters. Bayesian approach”, *ArXiv:physics/0409129* (2004)
- 300 [9] R. Brun and F. Rademakers, “ROOT - An object oriented data
301 analysis framework”, *Nucl. Inst. Meth.* **A389** (1997), 81 (see also
302 <http://root.cern.ch>)

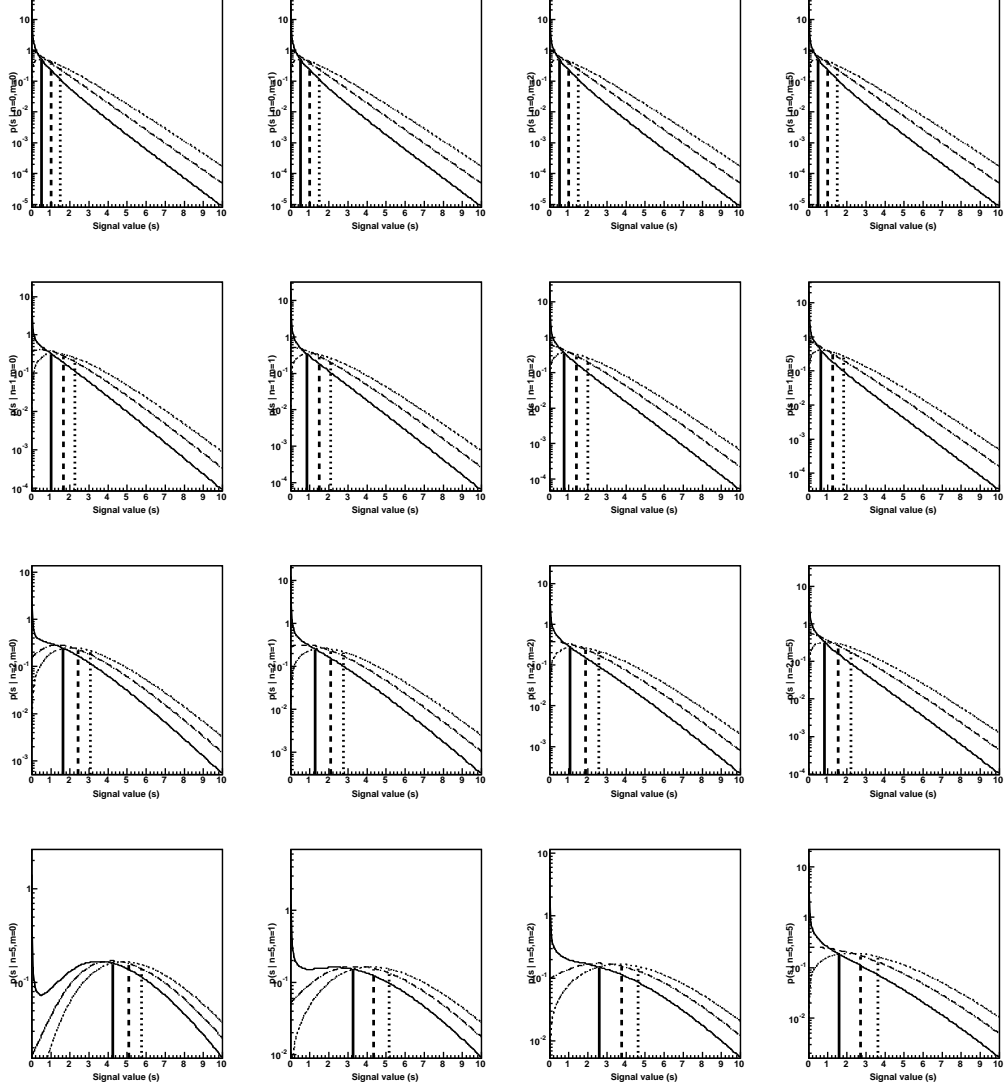


Figure 1: The plots show the posterior p.d.f.s for the signal $p(s|n, m)$ for different values of n, m and α in the case $c = 1$. Going from top to bottom, the plots correspond to $n = 0, 1, 2, 5$; going from left to right the plots correspond to $m = 0, 1, 2, 5$. The p.d.f.s have been evaluated assuming $\alpha = 0.5$ (continuous lines), $\alpha = 0$ (dashed lines) and $\alpha = -0.5$ (dotted lines). In each plot three vertical lines are also drawn with the same styles specified above, each indicating the average value of s from the corresponding p.d.f.s.

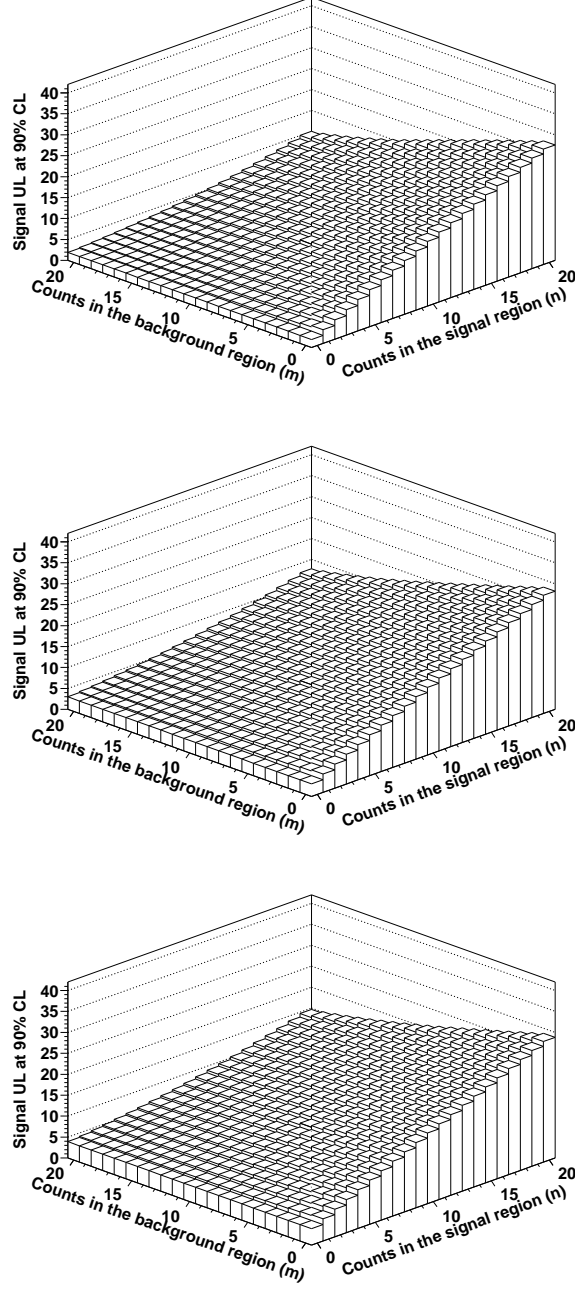


Figure 2: The plots show the values of the signal upper limits at 90% confidence level as a function of the counts observed in the signal and background regions, n and m , in the case $c = 1$. Going from top to bottom, the plots correspond to $\alpha = 0.5$, $\alpha = 0$ and $\alpha = -0.5$.

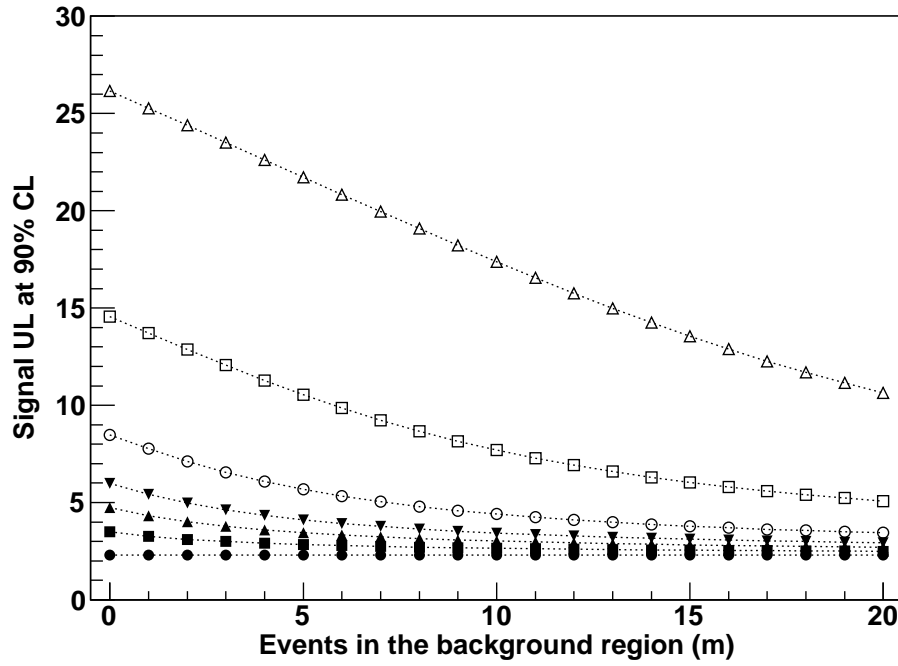


Figure 3: The plot shows the values of the signal upper limits at 90% confidence level as a function of the counts observed in the background region, m , in the case $c = 1$ and $\alpha = 0$, for some different values of the counts in the signal region. The calculation has been performed for $n = 0$ (●), 1 (■), 2 (▲), 3 (▼), 5 (○), 10 (□) and 20 (△).

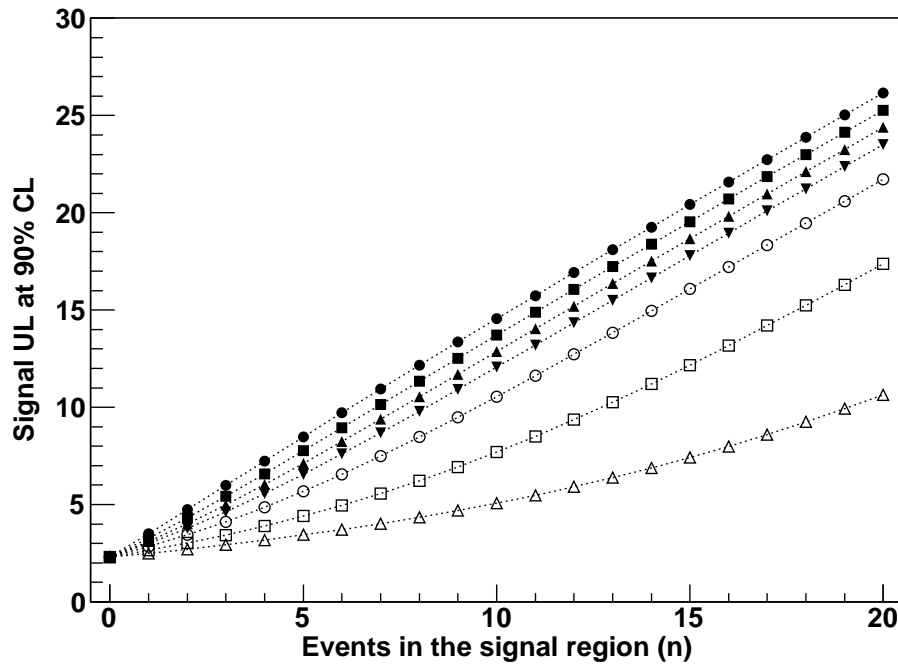


Figure 4: The plot shows the values of the signal upper limits at 90% confidence level as a function of the counts observed in the signal region, n , in the case $c = 1$ and $\alpha = 0$, for some different values of the counts in the background region. The calculation has been performed for $m = 0$ (●), 1 (■), 2 (▲), 3 (▼), 5 (○), 10 (□) and 20 (△).

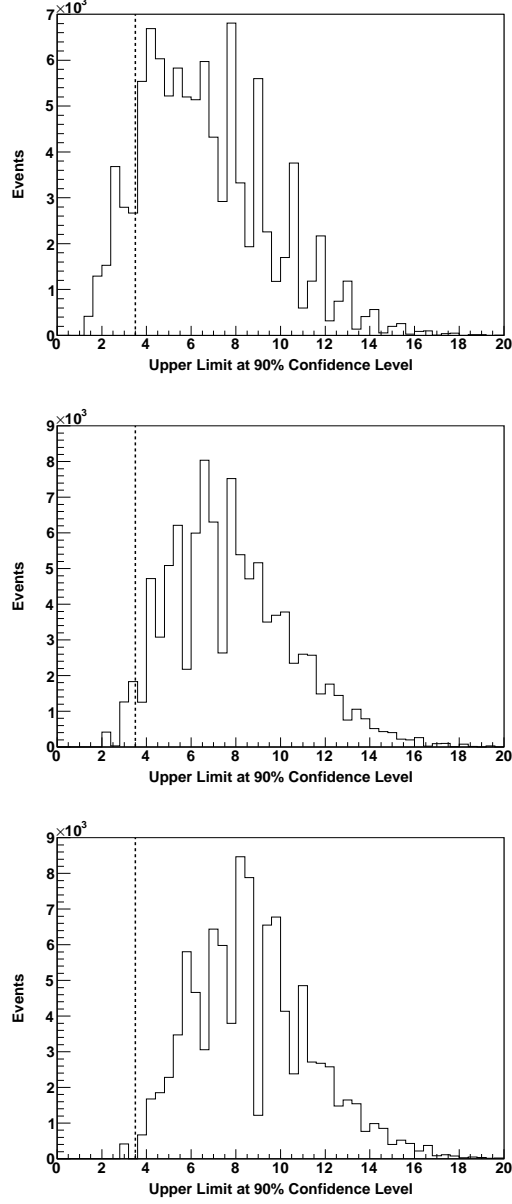


Figure 5: The plots show the distributions of the signal upper limits at 90% confidence level obtained from a sample of 10^5 simulated experiments with $s = 3.5$, $b = 2$ and $c = 1$. Going from top to bottom, the plots correspond to the upper limits evaluated by setting $\alpha = 0.5$, $\alpha = 0$ and $\alpha = -0.5$. The dashed lines indicate the true value of the signal. The coverage is graphically represented by the area of the histogram at the right of the dashed line normalized to the total area.

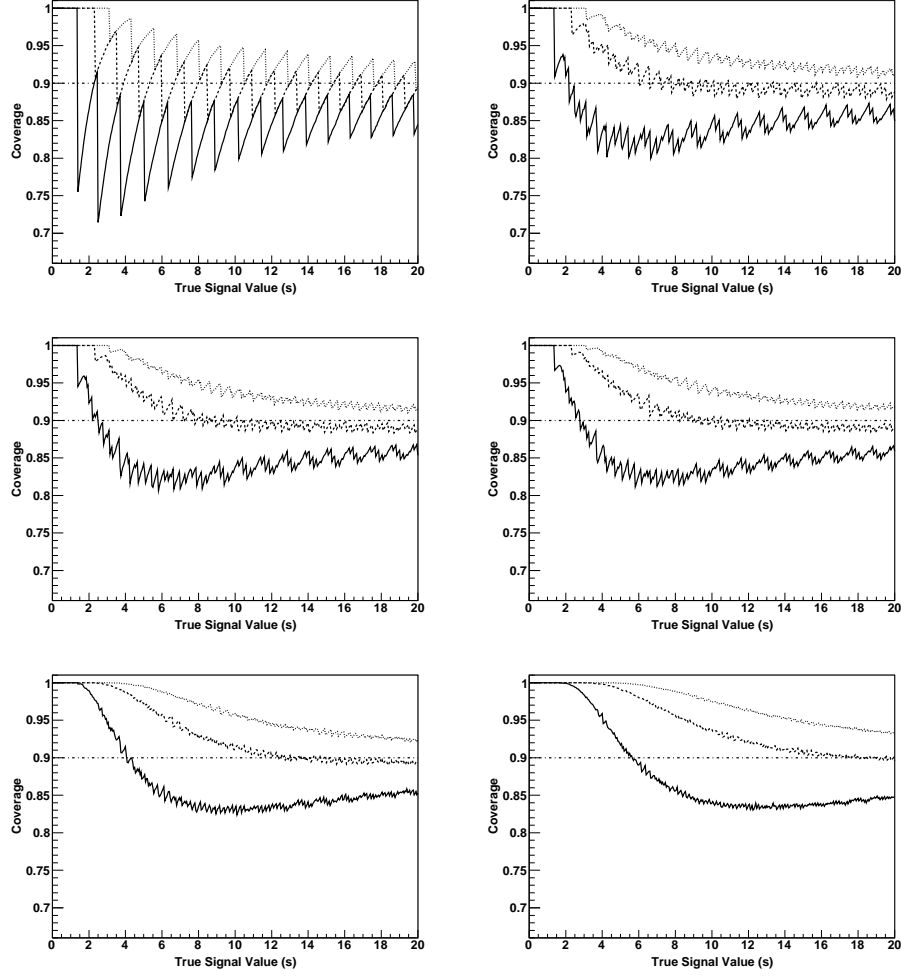


Figure 6: The plots show the frequentist coverage of the upper limits at 90% confidence level as a function of the true signal value s for different values of b with $c = 1$. From top left to bottom right the plots correspond to the cases $b = 0$, $b = 1$, $b = 1.5$, $b = 2$, $b = 5$ and $b = 10$. The upper limits have been evaluated by setting $\alpha = 0.5$ (continuous lines), $\alpha = 0$ (dashed lines) and $\alpha = -0.5$ (dotted lines). The dash-dotted lines indicate the 90% coverage.